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The Triangulation Calculus

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Abstract

The Duration Calculus is an important tool in the tool bag of the real-time systems engineer. The distinguishing and powerful feature of the Duration Calculus is interpretations are given with respect to (open) intervals of the real-line. This allows properties of intervals to be specified and reasoned about.

In an attempt to understand the relationship between the logic underlying the Duration Calculus and the domain over which it works, we have produced the Triangulation Calculus which (we claim) does for plane triangles what the Duration Calculus does for intervals.

The paper is very closely structured along the lines of [CRH93]. Readers familiar with that paper will recognize this, and have no problem making the translation.

1 Introduction

The Duration Calculus (DC, [CRH93]) is a logic for reasoning about intervals of the real line. This makes it suitable for specifying and reasoning about continuous and, in particular, hybrid systems — systems which combine discrete and continuous subsystems together.

The essence of the DC is the chop operator, $;$ (first to be found in the Interval Temporal Logic of [Moz85]). It provides for the interpretation of duration formulae over intervals by allowing an interval to be split into two. For example, the duration formula

$$D_1 ; D_2$$

is true over an interval (c, d) if and only if there is a mid-point m such that D_1 holds over (c, m) and D_2 holds over (m, d) .

1.1 Intervals are open

The definition of an interval of the Duration Calculus is that of an open interval. This has ramifications for the logic in that a formula that holds over two contiguous open intervals may not hold of the interval containing the union of those intervals whereas, of course, the opposite does hold (For the Triangulation Calculus version, see Theorem 1 and its Corollary).

1.2 Modalities

Modalities can be defined in terms of $;$. For instance, we may define $\diamond D$ — meaning D holds of some sub-interval — by the following definition:

$$\diamond D \stackrel{\text{def}}{=} true ; D ; true$$

and $\square D$ — over all subintervals D holds — from the dual

$$\square D \stackrel{\text{def}}{=} \neg \diamond \neg D \quad .$$

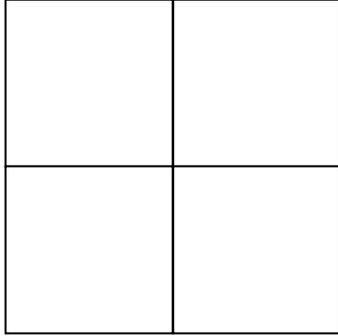


Figure 1: A partitioned square

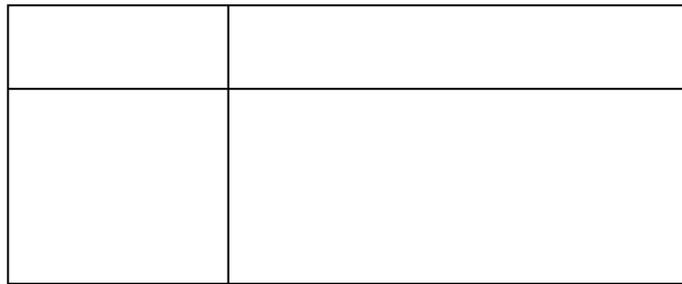


Figure 2: A partitioned rectangle

1.3 A geometric interpretation

Stepping back from the important continuous applications of the DC, we may look at its geometric significance in its manipulation of intervals. Essentially, the duration calculus allows properties of intervals to be addressed from within a logic. Moreover, the important property of an interval which provides the basis for the $;$ operator is that any non-trivial interval is decomposable into two disjoint subintervals which partition it.

This is a property of other geometric figures as well. For instance, a square is partitionable into four other squares all of equal size (see Figure 1); more generally, a rectangle is partitionable into a number of sub-rectangles (see Figure 2).

The example that we shall explore in this paper is that of the plane triangle, and its partitioning into four triangles arranged as shown in Figure 3.

2 The Triangulation Calculus

This section defines the syntax and semantics of the Triangulation Calculus (TC) as a first order logic with triangulation operators whose role is to decompose plane triangles.

The semantics of the TC is based on a theory of real valued functions on the complex plane.

An *interpretation* \mathcal{I} is an assignment of a function $f_{\mathcal{I}} : \mathcal{C} \rightarrow \mathcal{R}$ to each function symbol of the theory, and an assignment $b_{\mathcal{I}} : \mathcal{C} \rightarrow \{0, 1\}$ to each boolean state (i.e., a boolean valued function on the complex plane) b .

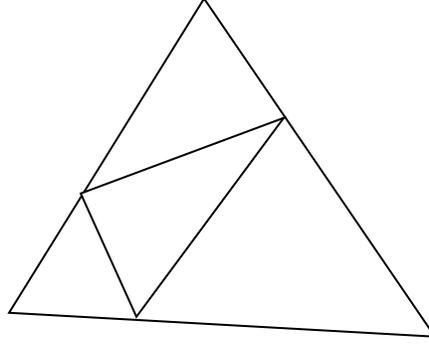


Figure 3: A partitioned triangle

2.1 Syntax

We assume that a mathematical theory MT has been given, i.e., we have states s and predicates P defined. From this we derive the syntax of the Triangulation Calculus.

The *triangulation terms* tt are defined as:

$$\begin{array}{l}
 tt \stackrel{\text{def}}{=} \quad a \quad \text{area} \\
 \quad \quad \quad | \quad x \quad \text{global variable} \\
 \quad \quad \quad | \quad \oplus(tt_1, \dots, tt_n) \quad n\text{-ary operator of the mathematical theory}
 \end{array}$$

The atomic *triangulation formulae* are built from duration terms and assertions P of the mathematical theory by the rule:

$$\begin{array}{l}
 D \stackrel{\text{def}}{=} \quad \triangleleft(tt_1, \dots, tt_n) \quad \triangleleft \text{ is an } n\text{-ary relational operator} \\
 \quad \quad \quad | \quad \underline{P} \quad \text{subtriangle}
 \end{array}$$

Atomic triangulation formulas are formed into triangulation formulas using the propositional connectives, the ‘trop’ operator¹, and universal quantification:

$$\begin{array}{l}
 D \stackrel{\text{def}}{=} \quad \neg D \quad \text{negation} \\
 \quad \quad \quad | \quad D_1 \wedge D_2 \quad \text{conjunction} \\
 \quad \quad \quad | \quad \frac{D_1/D_3 \setminus D_2}{D_4} \quad \text{trop operator} \\
 \quad \quad \quad | \quad \forall x \bullet D \quad \text{universal quantification}
 \end{array}$$

where x is a global variable. As for the DC, the same symbols can be used in assertions and triangulation formulae. Since assertions appear within the $\underline{\Delta}$ operator, this does not lead to ambiguity.

3 Semantics

We may now define the truth value of an arbitrary triangulation formula D . The truth value of D depends on the interpretation \mathcal{I} chosen for the state and boolean state names, the value of the global variables, and the triangle T .

¹A mix of ‘triangle’ and ‘chop’.

A valuation is, therefore, a pair (\mathcal{V}, T) where $\mathcal{V} : \text{Var} \rightarrow \mathcal{R}(= \text{Val})$ is a value assignment associating a real value with each global variable.

The semantics of a triangulation term tt in an interpretation \mathcal{I} is a function:

$$\mathcal{I}(tt) : \text{Val} \rightarrow \mathcal{R}$$

define inductively on the structure of triangulation terms by:

$$\begin{array}{lll} \mathcal{I}(a)(\mathcal{V}, T) & \stackrel{\text{def}}{=} & \text{area}(T) \quad \text{area of the triangle} \\ \mathcal{I}(x)(\mathcal{V}, T) & \stackrel{\text{def}}{=} & \mathcal{V}(x) \quad \text{value of the global variable} \\ \mathcal{I}(\oplus(tt_1, \dots, tt_n))(\mathcal{V}, T) & \stackrel{\text{def}}{=} & \oplus(\mathcal{I}(tt_1)(\mathcal{V}, T), \dots, \mathcal{I}(tt_n)(\mathcal{V}, T)) \end{array}$$

The semantics of a duration formula D in an interpretation I is a function defined recursively, using the following shorthand:

$$\begin{array}{ll} \mathcal{I}, \mathcal{V}, T \models D & \stackrel{\text{def}}{=} \quad \mathcal{I}(D)(\mathcal{V}, T) = tt \\ \mathcal{I}, \mathcal{V}, T \not\models D & \stackrel{\text{def}}{=} \quad \mathcal{I}(D)(\mathcal{V}, T) = ff \end{array}$$

such that

$$\begin{array}{ll} \mathcal{I}, \mathcal{V}, T \models \triangleleft(dt_1, \dots, dt_n) & \iff \triangleleft(\mathcal{I}(dt_1)(\mathcal{V}, T), \dots, \mathcal{I}(dt_n)(\mathcal{V}, T)) \\ \mathcal{I}, \mathcal{V}, T \models \underline{P} & \iff \text{area}(T) \neq 0 \wedge P_{\mathcal{I}}(t) = 1 \quad \forall t \in T \\ \mathcal{I}, \mathcal{V}, T \models \neg D & \iff \mathcal{I}, \mathcal{V}, T \not\models D \\ \mathcal{I}, \mathcal{V}, T \models D_1 \wedge D_2 & \iff \mathcal{I}, \mathcal{V}, T \models D_1 \wedge \mathcal{I}, \mathcal{V}, T \models D_2 \\ \mathcal{I}, \mathcal{V}, T \models \frac{D_1/D_3 \setminus D_2}{D_4} & \iff \exists T_1, T_2, T_3, T_4 \bullet T = \frac{T_1/T_3 \setminus T_2}{T_4} \wedge \mathcal{I}, \mathcal{V}, T_i \models D_i \\ & \quad i = 1, 2, 3, 4 \\ \mathcal{I}, \mathcal{V}, T \models \forall x \bullet D & \iff \mathcal{I}, \mathcal{V}', T \models D \\ & \quad \text{for all valuations } \mathcal{V}' \text{ which agree with } \mathcal{V} \text{ except at } x \end{array}$$

3.1 Satisfaction and Validity

A triangulation formula D is true in the interpretation \mathcal{I} ($\mathcal{I} \models D$) iff

$$\mathcal{I}, \mathcal{V}, T \models D$$

for every valuation (\mathcal{V}, T) . We also say that \mathcal{I} satisfies D . Furthermore, D is valid ($\models D$) iff D is true in every interpretation.

4 Abbreviations

Much of the power of the Duration Calculus is in the abbreviated notation that it provides for often used constructions and simple expressions. Such abbreviations provide:

- useful abbreviated representations for specification in terms of which intuition can be advanced;
- targets for which useful properties of the logic can proven.

A knowledge of which constructs are usefully abbreviated comes from working with a logic and, thus, from working with case studies for which the logic was created. We are in the (un)fortunate position of not having case studies for which this is the case, and so must look to other sources for our abbreviations. Of course, as with the remainder of this paper, we will provide abbreviations by analogy to the DC and define the following (as always, following [CRH93]):

The abbreviations $\vee, \implies, \Leftrightarrow, \exists$ are as would be expected (note \triangleleft binds more tightly than any of these):

$$\begin{aligned} \mathcal{I}, \mathcal{V}, T \models D_1 \vee D_2 &\iff \mathcal{I}, \mathcal{V}, T \not\models \neg D_1 \wedge \neg D_2 \\ \mathcal{I}, \mathcal{V}, T \models D_1 \implies D_2 &\iff \mathcal{I}, \mathcal{V}, T \models \neg D_1 \vee D_2 \\ \mathcal{I}, \mathcal{V}, T \models D_1 \Leftrightarrow D_2 &\iff \mathcal{I}, \mathcal{V}, T \models D_1 \implies D_2 \wedge D_2 \implies D_1 \\ \mathcal{I}, \mathcal{V}, T \models \exists x \bullet D &\iff \mathcal{I}, \mathcal{V}, T \not\models \forall x \bullet \neg D_1 \end{aligned}$$

Cascading quantifiers are also allowed, for example, $\forall x_1, x_2 \bullet D$, with their expected meaning.

It is notable that, because of the tighter binding of the \triangleleft operator, we can say nothing about the partitioning of the left- and right-hand sides of a triangle in the following formula:

$$\frac{P_1/P_3 \setminus P_2}{P_4} \Rightarrow \frac{P'_1/P'_3 \setminus P'_2}{P'_4} .$$

Constraining the partitioning is possible, as Theorem 1 shows.

Continuing with the abbreviations — logical constants can be derived:

$$\begin{aligned} true &\stackrel{\text{def}}{=} a \geq 0 \quad \text{a property of all (even degenerate) triangles} \\ false &\stackrel{\text{def}}{=} \neg true \end{aligned}$$

We distinguish degenerate triangles:

$$\triangleleft \stackrel{\text{def}}{=} a = 0$$

and may define modalities as:

$$\begin{aligned} \diamond D &\stackrel{\text{def}}{=} \frac{true/D \setminus true}{true} \\ \square D &\stackrel{\text{def}}{=} \neg \diamond \neg D \end{aligned}$$

5 Properties

Other than the properties of the functions upon which our logic is based, we see that the following properties of the Triangulation Calculus hold:

Monotonicity: If $\models D_i \Rightarrow D'_i$, $i = 1, 2, 3, 4$, then $\models \frac{D_1/D_3 \setminus D_2}{D_4} \Rightarrow \frac{D'_1/D'_3 \setminus D'_2}{D'_4}$. This is

False as ‘zero’: $\frac{D/false \setminus false}{false} \Rightarrow false$, as do $\frac{false/false \setminus D}{false}$, $\frac{false/D \setminus false}{false}$ and $\frac{false/false \setminus false}{D}$.

Point as ‘unit’:

$$D \iff \left(\frac{D/\underline{\Delta} \setminus \underline{\Delta}}{\underline{\Delta}} \right)$$

(remember: $\underline{\Delta} \iff a = 0$)

$$\iff \left(\frac{\underline{\Delta}/\underline{\Delta} \setminus D}{\underline{\Delta}} \right)$$

$$\iff \left(\frac{\underline{\Delta}/D \setminus \underline{\Delta}}{\underline{\Delta}} \right)$$

$$\iff \left(\frac{\underline{\Delta}/\underline{\Delta} \setminus \underline{\Delta}}{D} \right)$$

Disjunction over Trop

$$\frac{D_1 \vee D'_1/D_3 \setminus D_2}{D_4} \implies \frac{D_1/D_3 \setminus D_2}{D_4} \vee \frac{D'_1/D_3 \setminus D_2}{D_4}$$

$$\frac{D_1/D_3 \setminus D_2 \vee D'_2}{D_4} \implies \frac{D_1/D_3 \setminus D_2}{D_4} \vee \frac{D_1/D_3 \setminus D'_2}{D_4}$$

$$\frac{D_1/D_3 \vee D'_3 \setminus D_2}{D_4} \implies \frac{D_1/D_3 \setminus D_2}{D_4} \vee \frac{D_1/D'_3 \setminus D_2}{D_4}$$

$$\frac{D_1/D_3 \setminus D_2}{D_4 \vee D'_4} \implies \frac{D_1/D_3 \setminus D_2}{D_4} \vee \frac{D_1/D_3 \setminus D_2}{D'_4}$$

Exists over Trop

$$\frac{(\exists x \bullet D_1)/D_3 \setminus D_2}{D_4} \iff \exists x \bullet \frac{D_1/D_3 \setminus D_2}{D_4} \quad x \notin FV(D_i), i \neq 1$$

$$\frac{D_1/D_3 \setminus (\exists x \bullet D_2)}{D_4} \iff \exists x \bullet \frac{D_1/D_3 \setminus D_2}{D_4} \quad x \notin FV(D_i), i \neq 2$$

$$\frac{D_1/(\exists x \bullet D_3) \setminus D_2}{D_4} \iff \exists x \bullet \frac{D_1/D_3 \setminus D_2}{D_4} \quad x \notin FV(D_i), i \neq 3$$

$$\frac{D_1/D_3 \setminus D_2}{(\exists x \bullet D_4)} \iff \exists x \bullet \frac{D_1/D_3 \setminus D_2}{D_4} \quad x \notin FV(D_i), i \neq 4$$

Area additivity

$$\frac{a = r/a = t \setminus a = s}{a = u} \Rightarrow a = r + s + t + u$$

Conjecture.

$$\models a = r + s + t + u \wedge t \geq a/4 \Rightarrow \frac{a = r/a = t \setminus a = s}{a = u}$$

This is not the case when $t < a/4$, from the well-known theorem.

5.1 Axioms of $\underline{\underline{P}}$ and $\underline{\Delta}$

$\underline{\underline{1}}$ If 1 is the everywhere 1 state, then

$$\models \underline{\underline{1}} \iff (a > 0)$$

$\underline{0}$ If 0 is the everywhere 0 state, then

$$\models \underline{0} \Leftrightarrow \text{false}$$

\underline{P}

$$\models \underline{P} \Rightarrow \Box(\underline{\wedge} \vee \underline{P})$$

As P holds on every point of the non-trivial triangle of the left-hand-side, then P holds of every point of any sub-triangle, and $\underline{\wedge}$ holds of non-empty sub-triangle of the right-hand-side. Note the importance of the $\text{area}(T) \neq 0$ in the definition of the interpretation of \underline{P} .

MT If $\vdash_{MT} P \Rightarrow Q$, i.e., P implies Q in MT then

$$\models \underline{P} \Rightarrow \underline{Q}$$

Conjunction

$$\models \underline{P} \wedge \underline{Q} \Rightarrow \underline{P \wedge Q}$$

Theorem 1.

$$\begin{aligned} &\models \underline{P} \wedge a = r + s + t + u \wedge t \geq a/4 \\ &\Rightarrow \frac{\underline{P} \wedge a=r / \underline{P} \wedge a=t \setminus \underline{P} \wedge a=s}{\underline{P} \wedge a=u} \end{aligned}$$

Corollary.

$$\models \underline{P} \Rightarrow \frac{\underline{P} / \underline{P} \setminus \underline{P}}{\underline{P}}$$

6 Conclusion

This paper has provided the beginning of a calculus of plane triangles, the Triangulation Calculus, based on the principles of the Duration Calculus. In it we have explored the relationship between the logic and the domain. In particular, we replaced the chop operator by the trop operator, and saw that the logical characterization of the trop operator was non-trivial in terms of the way that a (commutative) sum of four values could be ‘tropped’.

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References

- [CRH93] Zhou Chaochen, Anders P. Ravn, and Michael R. Hansen. An Extended Duration Calculus for Hybrid Real-Time Systems. Number 736 in Lecture Notes In Computer Science, pages 37–59. Springer-Verlag, 1993.
- [Moz85] Ben Mozskowski. A Temporal Logic for Multilevel Reasoning about Hardware. *IEEE Computers*, 18:10–19, 1985.